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# An improved iteration regularization method and application to reconstruction of dynamic loads on a plate

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ABSTRACT

We present an improved iteration regularization method for solving linear inverse problems. The algorithm considered here is detailedly given and proved that the computational costs for the proposed method are nearly the same as the Landweber iteration method, yet the number of iteration steps by the present method is even less. Meanwhile, we obtain the optimum asymptotic convergence order of the regularized solution by choosing a posterior regularization parameter based on Morozov's discrepancy principle, and the present method is applied to the identification of the multi-source dynamic loads on a surface of the plate. Numerical simulations of two examples demonstrate the effectiveness and robustness of the present method.



## 1. Introduction

Inverse problems have arisen in lots of application fields ranging from scientific computation to many practical engineering problems [1–13]. A significant amount of research work has attracted much attention as a result of their various applications [14–22]. However, in many practical engineering problems, it is impossible or very difficult to directly measure the dynamic load in real time. It means that the dynamic load must be recovered with the help of indirect measurements. For instance, Zhang and Mann III estimated the structural intensity and the force distribution for plates by the FFT method [23,24]. The structural intensity for plates was calculated by Nedjade and Singh in terms of Spatial Fourier Transforms and dynamic response [25]. Dynamic force identification based on enhanced least squares and total least squares schemes was studied in [26]. Liu and Han presented an inverse procedure for identifying both concentrated and extended line load using Green's function and Heaviside step function in time domain [27,28]. Unfortunately, these inverse problems mentioned above are complex and inherently ill-posed. From the view of mathematical theory, any direct numerical treatment of ill-posed problems will generate not useful solutions which cause large deviations from the exact solutions of the large scale inverse problems. So far, many regularization methods have been developed to solve these ill-posed phenomena [29,30]. Nevertheless, for the large scale inverse problems, the preferred one is the iterative regularization method. In Ref. [31], Neubauer assured that the Landweber iteration method is a regularization method, and has a good stability when solving the large disturbed ill-posed problems, yet the rate of convergence of regularized solution by this method is very slow and inefficient. In order to solve these problems, in this paper, we propose an improved iteration regularization method, investigate the minimum of this minimization problem, and apply it to reconstruct distributed dynamic loads on a plate by its steady-state responses.

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This paper is organized as follows. In Section 2 we establish an improved iteration regularization method and prove its regular property. In Section 3 we prove that this method can obtain the optimum asymptotic convergence order of the regularized solution provided that the regularization parameter is chosen by the appropriate posterior parameter rule. In Section 4 we generalize our results to the iteration scheme and demonstrate the effectiveness of the proposed method using a numerical test, then apply this method to identify the multi-source dynamic loads acting on a plate in Section 5 and we give a conclusion in Section 6.

## 2. The establishment of iteration regularization method

Let $X$ and $Y$ be real Hilbert spaces and $K \in L(X, Y)$, i.e., $K : X \to Y$ is a bounded linear operator. We consider the equation

$$Kx = y \quad (y \in R(K)). \tag{2.1}$$

Throughout this paper we assume:

(H1) $y^\delta \in X$ is the available noisy data with

$$\|y - y^\delta\| \le \delta$$

and known noise level $\delta$.

In practice, instead of (2.1) we usually have a perturbed equation

$$Kx = y^\delta. \tag{2.2}$$

In general terms, problem (2.2) is ill-posed. Regularization is the approximation of an ill-posed problem by a family of neighboring well-posed problems. For obtaining the stable solutions of linear ill-posed problems, we have to use a regularization method.

(H2) Let $K : X \to Y$ be a biunivocal compact operator and $y \in R(K)$.

Let $(\mu_j, x_j, y_j)_{j \in N}$ be a singular system for the linear operator $K : X \to Y$, $\mu_1 \ge \mu_2 \ge \cdots \ge \mu_j \ge \mu_{j+1} \ge \cdots > 0$. It is easy to check that under the condition of (H2) Eq. (2.1) has a unique solution $x$. Exploiting the singular system, we obtain

$$x = \sum_{j=1}^{\infty} \frac{1}{\mu_j} (y, y_j) x_j.$$

Define operator $R_n$:

$$R_n = \left[ \sum_{j=0}^{n-1} \mathrm{e}^{-(wK^*K)^{\frac{1}{a}} j} (I - \mathrm{e}^{-(wK^*K)^{\frac{1}{a}}}) \right]^a (K^*K)^{-1} K^*, \tag{2.3}$$

where $a$ is a positive integer constant and $0 < w < \frac{1}{\|K^*K\|}$ is a relaxation factor.

**Theorem 2.1.** *If $y \in R(K)$, then*

$$R_n y = \sum_{j=1}^{\infty} \frac{q(n, \mu_j)}{\mu_j} (y, y_j) x_j, \tag{2.4}$$

$$q(n, \mu) = \left[ 1 - \mathrm{e}^{-(w\mu^2)^{\frac{1}{a}} n} \right]^a, \tag{2.5}$$

*where $a$ is a positive integer constant.*

**Proof.** Since $y \in R(K)$,

$$K^*Kx = K^*y.$$

Due to

$$\begin{aligned}
R_n y &= \left[ \sum_{j=0}^{n-1} \mathrm{e}^{-(wK^*K)^{\frac{1}{a}} j} (I - \mathrm{e}^{-(wK^*K)^{\frac{1}{a}}}) \right]^a (K^*K)^{-1} K^* y \\
&= \left[ \sum_{j=0}^{n-1} \mathrm{e}^{-(wK^*K)^{\frac{1}{a}} j} (I - \mathrm{e}^{-(wK^*K)^{\frac{1}{a}}}) \right]^a (K^*K)^{-1} K^* K x \\
&= \left[ \sum_{j=0}^{n-1} \mathrm{e}^{-(wK^*K)^{\frac{1}{a}} j} \left( I - \mathrm{e}^{-(wK^*K)^{\frac{1}{a}}} \right) \right]^a \sum_{i=1}^{\infty} \frac{(y, y_i)}{\mu_i} (K^*K)^{-1} K^* K x_i \\
&= \sum_{i=1}^{\infty} \frac{(y, y_i)}{\mu_i} \left[ \sum_{j=0}^{n-1} \mathrm{e}^{-(wK^*K)^{\frac{1}{a}} j} \left( I - \mathrm{e}^{-(wK^*K)^{\frac{1}{a}}} \right) \right]^a (K^*K)^{-1} K^* K x_i,
\end{aligned}$$

we can obtain

$$R_n y = \sum_{j=1}^{\infty} \frac{q(n,\mu_j)}{\mu_j}(y, y_j)x_j. \quad \square$$

**Theorem 2.2.** *If $q(n,\mu)$ is given by (2.5), we have the following results:*

(i) $0 < q(n,\mu) < 1$;
(ii) $q(n,\mu) < q(n+1,\mu)$, $\lim_{n\to\infty} q(n,\mu) = 1$;
(iii) $q(n,\mu) < w^{1/2}n^{a/2}\mu$;
(iv) $1 - q(n,\mu) \le a e^{-(w\mu^2)^{\frac{1}{a}}n}$.

**Proof.** Exploiting

$$0 < w < \frac{1}{\|K^*K\|}$$

and

$$0 < w\mu^2 < 1,$$

we obtain

$$0 < \mathrm{e}^{-(w\mu^2)^{\frac{1}{a}}n} < 1,$$

then it gives the assertion.

(ii) Obviously, the second result of Theorem 2.2 is also valid.

(iii) Since

$$\frac{1-x}{1+x} < \mathrm{e}^{-2x}, \quad x > 0,$$

we have

$$q(n,\mu) < \left[\frac{2(w\mu^2)^{\frac{1}{a}}n}{2+(w\mu^2)^{\frac{1}{a}}n}\right]^a \le w\mu^2 n^a.$$

Exploiting $0 < q(n,\mu) < 1$, we obtain

$$q(n,\mu) < \sqrt{q(n,\mu)} < w^{1/2}n^{a/2}\mu.$$

(iv) By virtue of Bernoulli inequality

$$\left[1 - \mathrm{e}^{-(w\mu^2)^{\frac{1}{a}}n}\right]^a \ge 1 - a\mathrm{e}^{-(w\mu^2)^{\frac{1}{a}}n},$$

then

$$1 - q(n,\mu) \le a\mathrm{e}^{-(w\mu^2)^{\frac{1}{a}}n}.$$

It is well known that a regularization method consists of a regularization operator and a parameter choice rule which is convergent in the sense that if the regularization parameter is chosen according to that rule, then the regularized solutions converge to the true solution in the norm as the noise level tends to zero. It follows from Theorem 2.2 that $q(n,\mu)$ is a regularizing filter operator. When $n \to \infty$, $R_n y \to x$. Herein, $n$ rule is as a regularization parameter. Choosing $y^\delta \in R(k)$ yields that $x_n^\delta = R_n y^\delta$ is an approximate solution of (2.1). $\square$

**Theorem 2.3.** (i) $\|R_n\| \le w^{1/2}n^{a/2}$. (ii) $\|R_n y^\delta - x\| \le w^{1/2}n^{a/2}\delta + \|R_n y - x\|$.

**Proof.** (i)

$$\|R_n y\|^2 = \left\|\sum_{j=1}^{\infty}\frac{q(n,\mu_j)}{\mu_j}(y,y_j)x_j\right\|^2 = \sum_{j=1}^{\infty}\left|\frac{q(n,\mu_j)}{\mu_j}\right|^2 |(y,y_j)|^2,$$

which together with Theorem 2.2 gives

$$\|R_n y\|^2 \le w n^a \sum_{j=1}^{\infty}|(y,y_j)|^2 \le w n^a \|y\|^2.$$

Then the proof is complete.

(ii)

$$\begin{aligned}\|R_n y^\delta - x\| &\le \|R_n y^\delta - R_n y\| + \|R_n y - x\| \\ &\le \|R_n\|\delta + \|R_n y - x\| \\ &\le n^{a/2} w^{1/2}\delta + \|R_n y - x\|. \quad \square\end{aligned}$$

## 3. The posterior parameter choice and convergence

Throughout the whole section we use the notations which have been introduced in Section 2. In the following we will discuss Morozov's discrepancy principle for choosing the regularization parameter $n$.

**Lemma 3.1.** *Let $P : Y \to R(K)$ be the orthogonal projector onto $\overline{R(K)}$, $y^\delta \in R(K)$. Set $\rho(n) = \|KR_n y^\delta - y^\delta\|$. Then*

$$\|KR_n y - y\|^2 = \sum_{j=1}^{\infty} |1 - q(n, \mu_j)|^2 |(y, y_j)|^2 + \|Py - y\|^2. \tag{3.1}$$

**Proof.**

$$\begin{aligned}\|KR_n y - y\|^2 &= \|KR_n y - Py + Py - y\|^2 \\ &= \|KR_n y - Py\|^2 + \|Py - y\|^2 \\ &= \left\| K\left(\sum_{j=1}^{\infty} \frac{q(n, \mu_j)}{\mu_j}(y, y_j)x_j\right) - K\sum_{j=1}^{\infty}\frac{1}{\mu_j}(y, y_j)x_j\right\|^2 + \|Py - y\|^2 \\ &= \sum_{j=1}^{\infty} |1 - q(n, \mu_j)|^2 |(y, y_j)^2| + \|Py - y\|^2.\end{aligned}$$

Moreover, it is easy to check that $\rho(n+1) < \rho(n)$ and $\rho(n) \to \|Py^\delta - y^\delta\|$ as $n \to \infty$. $\square$

**Lemma 3.2.** *Let $\tau > 1$ and* (H1)–(H2) *be satisfied. Choose the regularization parameter $n = n(\delta)$, such that $\|KR_n y^\delta - y^\delta\| \le \tau\delta$ occurs for the first time. Then*

$$\|KR_{n-1} y - y\| \ge (\tau - 1)\delta. \tag{3.2}$$

**Proof.** Obviously,

$$\|KR_{n-1} y^\delta - y^\delta\| > \tau\delta.$$

It follows from Lemma 3.1 that

$$\begin{aligned}\|(KR_{n-1} - I)(y - y^\delta)\|^2 &= \sum_{j=1}^{\infty} |1 - q(n-1, \mu_j)|^2 |(y - y^\delta, y_j)|^2 + \|P(y - y^\delta) - (y - y^\delta)\|^2 \\ &\le \sum_{j=1}^{\infty} |(y - y^\delta, y_j)|^2 + \|(P - I)(y - y^\delta)\|^2 \\ &\le \|P(y - y^\delta)\|^2 + \|(P - I)(y - y^\delta)\|^2 \\ &\le \delta^2.\end{aligned}$$

Then we have

$$\begin{aligned}\|KR_{n-1} y - y\| &= \|KR_{n-1} y^\delta - y^\delta - (KR_{n-1} - I)(y^\delta - y)\| \\ &\ge \|KR_{n-1} y^\delta - y^\delta\| - \|(KR_{n-1} - I)(y^\delta - y)\| \\ &\ge (\tau - 1)\delta.\end{aligned}$$

Now the assertion can be proved easily. $\square$

**Theorem 3.1.** *Let $x = (K^*K)^v z,\ z \in X$, and $\|z\| \le E$. Choose the regularization parameter $n = n(\delta)$, such that*

$$\|KR_n y^\delta - y^\delta\| \le \tau\delta, \quad \tau > 1$$

*occurs for the first time. If $\overline{R(K)} = Y$, then*

(i) $n = O\left(\delta^{-\frac{2}{(2v+1)a}}\right)$;

(ii) $\|R_n y^\delta - x\| = O\left(\delta^{\frac{2v}{2v+1}}\right)$.

**Proof.** (i)

$$\begin{aligned}\|KR_n y - y\|^2 &= \sum_{j=1}^{\infty} |1 - q(n, \mu_j)|^2 |(y, y_j)|^2 \\ &= \sum_{j=1}^{\infty} |1 - q(n, \mu_j)|^2 |(y, Tx_j)|^2 \\ &= \sum_{j=1}^{\infty} \mu_j^{4v+2} |1 - q(n, \mu_j)|^2 |(z, x_j)|^2.\end{aligned}$$

Exploiting

$$t^x (1-t)^{k-1} \le x^x k^{-x}, \quad k \ge 2,\ 0 \le t \le 1,\ x \ge 1$$

and

$$\mathrm{e}^{\frac{-x}{1-x}} < 1 - x, \quad x < 1,\ x \ne 0,$$

we obtain

$$\begin{aligned}\mu^{2v+1} |1 - q(n-1, \mu)| &\le \mu^{2v+1} a \mathrm{e}^{-(w\mu^2)^{1/a}(n-1)} \\ &= \left[(w\mu^2)^{1/a}\right]^{\frac{2v+1}{2}a} \mathrm{e}^{-(w\mu^2)^{1/a}(n-1)} w^{-\frac{2v+1}{2}} a \\ &\le \left[\frac{(w\mu^2)^{1/a}}{(w\mu^2)^{1/a}+1}\right]^{\frac{2v+1}{2}a} \left[1 - \frac{(w\mu^2)^{1/a}}{(w\mu^2)^{1/a}+1}\right]^{n-1} w^{-\frac{2v+1}{2}} a \left[(w\mu^2)^{1/a}+1\right]^{\frac{2v+1}{2}a} \\ &\le 2^{\frac{2v+1}{2}a} a w^{-\frac{2v+1}{2}} \left[\frac{(w\mu^2)^{1/a}}{(w\mu^2)^{1/a}+1}\right]^{\frac{2v+1}{2}a} \left[1 - \frac{(w\mu^2)^{1/a}}{(w\mu^2)^{1/a}+1}\right]^{n-1} \\ &\le 2^{\frac{2v+1}{2}a} a w^{-\frac{2v+1}{2}} \left(\frac{2v+1}{2}a\right)^{\frac{2v+1}{2}a} n^{-\frac{2v+1}{2}a}.\end{aligned} \tag{3.3}$$

It follows from Lemma 3.2 and (3.3) that

$$(\tau - 1)\delta \le \|KR_{n-1} y - y\| \le aE 2^{\frac{2v+1}{2}a} w^{-\frac{2v+1}{2}} \left(\frac{2v+1}{2}a\right)^{\frac{2v+1}{2}a} n^{-\frac{2v+1}{2}a}.$$

Then we can obtain

$$n \le a(2v+1) w^{-1/a} \left[\frac{Ea}{(\tau-1)\delta}\right]^{\frac{2}{(2v+1)a}}. \tag{3.4}$$

(ii)

$$\begin{aligned}\|R_n y - x\|^2 &= \sum_{j=1}^{\infty} |1 - q(n, \mu_j)|^2 |(x, x_j)|^2 \\ &= \sum_{j=1}^{\infty} |1 - q(n, \mu_j)|^2 |((K^*K)^v z, x_j)|^2 \\ &= \sum_{j=1}^{\infty} \mu_j^{4v} |1 - q(n, \mu_j)|^2 |(z, x_j)|^2.\end{aligned}$$

Set $p = \frac{2v+1}{2v}$, $q = (2v+1)$. Due to Hölder inequality, we obtain

$$\sum_{j=1}^{\infty}\mu_j^{4v}|1-q(n,\mu_j)|^2|(z,x_j)|^2 \le \left[\sum_{j=1}^{\infty}\mu_j^{4v+2}|1-q(n,\mu_j)|^2|(z,x_j)|^2\right]^{1/p}\left[\sum_{j=1}^{\infty}|(z,x_j)|^2\right]^{1/q}$$
$$= \|KR_ny-y\|^{2/p}E^{2/q}.$$

Moreover,

$$\|KR_ny-y\| \le \|KR_ny^\delta-y^\delta\| + \|(KR_n-I)(y-y^\delta)\| \le (\tau+1)\delta,$$

then

$$\|R_ny-x\| \le E^{\frac{1}{2v+1}}(\tau+1)^{\frac{2v}{2v+1}}\delta^{\frac{2v}{2v+1}}, \tag{3.5}$$

which together with (3.4) and Theorem 2.3, gives

$$\|R_ny^\delta-x\| \le \left[[(2v+1)a]^{a/2}\left(\frac{Ea}{\tau-1}\right)^{\frac{1}{2v+1}} + E^{\frac{1}{2v+1}}(\tau+1)^{\frac{2v}{2v+1}}\right]\delta^{\frac{2v}{2v+1}}. \quad \square$$

## 4. Iterative algorithm and numerical studies

In this section we will generate our results of Sections 2 and 3 for the method of iteration regularization, and then demonstrate the effectiveness of this iteration regularization method by a numerical example test.

### 4.1. Iterative algorithm

First, we discuss the following iterative scheme

$$B_{n+1} = \mathrm{e}^{-(\omega K^*K)^{1/a}}B_n + I, \qquad B_1 = I.$$

It is easy to check

$$B_n = \sum_{j=0}^{n-1}\mathrm{e}^{-(\omega K^*K)^{1/a}j}.$$

Now we can give the regularized solution by

$$R_ny^\delta = \left[I-\mathrm{e}^{-(\omega K^*K)^{1/a}}\right]^aB_n^a(K^*K)^{-1}K^*y^\delta.$$

Then we can present the iterative algorithm as follows:

(i) Set $B_1 = I$;
(ii) compute $B_{n+1} = \mathrm{e}^{-(\omega K^*K)^{1/a}}B_n + I$;
(iii) compute $R_ny^\delta = \left[I-\mathrm{e}^{-(\omega K^*K)^{1/a}}\right]^aB_n^a(K^*K)^{-1}K^*y^\delta$, if $\|KR_ny^\delta-y^\delta\| \le \tau\delta$, then stop, or switch to (ii).

In fact, the iteration index $n$ plays the role of the regularization parameter $\alpha$, and the stopping rule performs the role of the parameter selection method. In addition, Theorem 3.1 shows that the present iteration regularization method yields order-optimal regularized solution and the greatly reduced number of iterations. Thus the present iteration regularization method with optimal rate of convergence of the approximation error is a very efficient candidate for regularizing inverse problems.

### 4.2. Benchmark test

Generally, for the ill-posed problems, inevitable small errors during the measurement can make the prediction of the expected solutions useless, which leads to numerical instabilities in the solutions. However, in practice, conventional numerical methods cannot deal with these problems. In order to overcome this difficulty, the Landweber iteration and Tikhonov regularization methods are usually employed as a stabilization technique in the study [5,31–33]. For the completeness of our study, we briefly summarize two commonly used regularization methods as follows. First, Tikhonov regularization is one of the most well-known regularization methods in solving ill-posed problems. In addition, the corresponding regularization technique was originally proposed independently by Tikhonov, and its convergence can be accelerated when regularizing in norms is stronger than the usual norm in Hilbert space. Its major difficulty lies in the ways to find an optimal regularization parameter. A good regularization parameter will choose the optimal weight to obtain a right balance between the perturbation error and the regularization error in the regularized solution. Nevertheless, especially for large scale inverse problems, the Landweber iterative regularization method is usually an attractive alternative to the

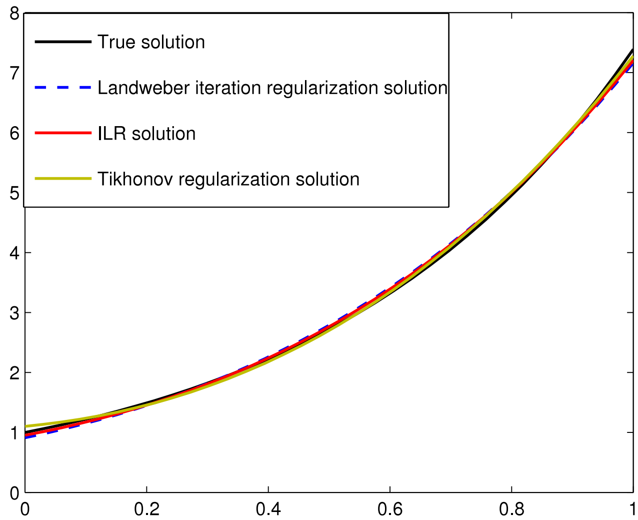


**Fig. 1.** Numerical results of Eq. (4.1).

**Table 1**
Identification errors of three different regularization methods.

| Noise level 0.0001 | | | |
|---|---|---|---|
| | Maximum error (%) | Average error (%) | Iterative number |
| Landweber iteration regularization method | 1.2004 | 0.6353 | 600 |
| Present method | 0.6706 | 0.4297 | 50 |
| Tikhonov regularization method | 1.3849 | 0.5863 | 0 |

Tikhonov regularization method and the asymptotic order can be improved through iterations. For these reasons above, the expected solution obtained by the present method (ILR) will be compared to the regularized solutions by two regularization methods mentioned above in the following numerical example.

To demonstrate the effectiveness of the proposed method, we consider the first kind of Fredholm integral equation as follows:

$$\int_0^1 \mathrm{e}^{ts} x(s) \mathrm{d}s = \frac{\mathrm{e}^{t+2} - 1}{t+2}, \quad t \in [0, 1]. \tag{4.1}$$

It is easy to check that the true solution of Eq. (4.1) is $x(s) = \mathrm{e}^{2s}$. In general terms, we usually consider the perturbed equation

$$\int_0^1 \mathrm{e}^{ts} x(s) \mathrm{d}s = y^\delta(t), \quad t \in [0, 1]. \tag{4.2}$$

Discretizing Eq. (4.2), we can obtain

$$\frac{1}{N} \sum_{j=1}^{N} e^{t_i s_j} x(s_j) = y_i^\delta, \quad i, j = 1, 2, \ldots, N, \tag{4.3}$$

where

$$t_i = \frac{i-1}{N}, \qquad s_j = \frac{j-1}{N}, \qquad y_i^\delta = y(t_i) + \theta_i \delta,$$

$\theta_i$ is a random number and satisfies $|\theta_i| \le 1$.

To analyze the performance of the present method, we choose the noisy level $\delta = 0.0001$, $N = 34$, $a = 2$, $\tau = 3$. Applying PC-MATLAB environment, we obtain the following results.

Fig. 1 indicates that the present method and two regularization methods mentioned above are stable and effective in identifying the true solution of the first kind of Fredholm integral equation (4.1). In fact, more informative results on the performances of these methods are shown in Table 1, from which it can be seen that the present method is more precise and effective than the Landweber iteration and Tikhonov regularization methods, and the numerically optimal convergence rate of the regularized solution roughly coincides with the theoretical analysis.

**Table 2**
The material properties of the plate.

| Material properties of the plate | | | | |
|---|---|---|---|---|
| Material constants | Elastic modulus | Poisson ratio | Density | Damping coefficient |
| | $7.0 \times 10^{10}$ N/m$^2$ | 0.33 | $2.8 \times 10^3$ kg/m$^3$ | 0.05 |

## 5. Application

To illustrate the present methodology for use in determining the unknown time-dependent multi-source dynamic loads acting on a plate, we need to know the following for a linear elastic structure.

Here we consider the multi-source dynamic load identification problem for a linear and time-invariant dynamic system. The response at an arbitrary receiving point in a structure can be expressed as a convolution integral of the forcing time history and the corresponding Green's kernel in time domain [27]:

$$y(t) = \int_0^t G(t-\tau)p(\tau)\mathrm{d}\tau, \tag{5.1}$$

where $y(t)$ is the response which can be displacement, velocity, acceleration, strain, etc. $G(t)$ is the corresponding Green's function, which is the kernel of impulse response. $p(t)$ is the desired unknown dynamic load acting on the structure.

By discretizing this convolution integral, the whole concerned time period is separated into equally spaced intervals, and Eq. (5.1) is transformed into the following system of algebraic equations:

$$Y(t) = G(t)P(t), \tag{5.2}$$

or equivalently,

$$\begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix} = \begin{pmatrix} g_1 & & & \\ g_2 & g_1 & & \\ \vdots & \vdots & \ddots & \\ g_m & g_{m-1} & \cdots & g_1 \end{pmatrix} \begin{pmatrix} p_1 \\ p_2 \\ \vdots \\ p_m \end{pmatrix} \triangle t,$$

where $y_i$, $g_i$, and $p_i$ are response, Green's function matrix and input force at time $t = i\triangle t$, respectively. $\triangle t$ is the discrete time interval. Since the structure without applied force is static before force is applied, $y_0$ and $g_0$ are equal to zero. All the elements in the upper triangular part of $G$ are zeros and are not shown. The special form of Green's function matrix reflects the characteristic of the convolution integral.

To recover the time history $P(t)$, the knowledge of $y(t)$ and $G(t)$ are required. In fact, the response at a receiving point and the numerical Green's function of a structure can be obtained by finite element method. However, the problem of identifying the dynamic load $P(t)$ by $y(t)$ and $G(t)$ is usually ill-posed, and cannot be solved by inverse matrix method. In the following, our method will be suggested to solve this problem.

A practical engineering problem is to determine vertical forces of a plate, which is shown in Fig. 2. The size of plate is 1200.0 mm in length, 500.0 mm in width, and 20.0 mm in thickness. Four supported beams are under the plate, and their sectional radius is 10.0 mm. The material properties of the plate are listed in Table 2.

The vertical concentrated load is applied to the outside surface, and the measured response is the vertical displacement. The bottoms of four supported beams are fixed, and the other parts of the plate are free. We establish its finite element model as shown in Fig. 2. The arrow in Fig. 2 denotes the action point of the dynamic force.

The concentrated loads are defined as follows:

$$F_1(t) = \begin{cases} q_1 \sin\left(\dfrac{2\pi t}{t_d}\right), & 0 \le t \le 2t_d \\ 0, & t < 0 \text{ and } t > 2t_d \end{cases}$$

$$F_2(t) = \begin{cases} 4q_2 t/t_d, & 0 \le t \le t_d/4 \\ 2q_2 - 4q_2 t/t_d, & t_d/4 < t \le 3t_d/4 \\ 4q_2 t/t_d - 4q_2, & 3t_d/4 < t \le t_d \\ 0, & t > t_d \end{cases}$$

where $t_d$ is the time cycle of the sine force, and $q_i$ $(i = 1, 2)$ is a constant amplitude of the force. When $t_d = 0.004$ s, $q_1 = 1000$ N, and $q_2 = 800$ N, the sine force and triangle force are shown in Figs. 3–4.

Herein, the experimental data of response is simulated by the computed numerical solution, and the corresponding vertical displacement response can be obtained by finite element method. Furthermore, a noise is directly added to the computer-generated response to simulate the noise-contaminated measurement, and the noisy response is defined as follows:

$$Y_{\mathrm{err}} = Y_{\mathrm{cal}} + l_{\mathrm{noise}} \cdot \mathrm{std}(Y_{\mathrm{cal}}) \cdot \mathrm{rand}(-1, 1),$$

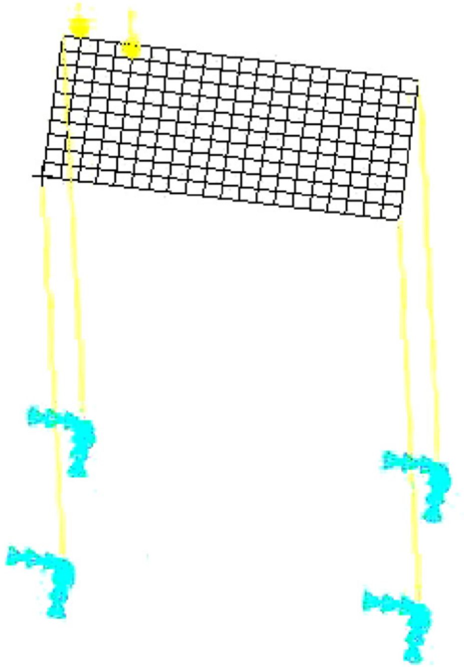

**Fig. 2.** The finite element model of the plate.

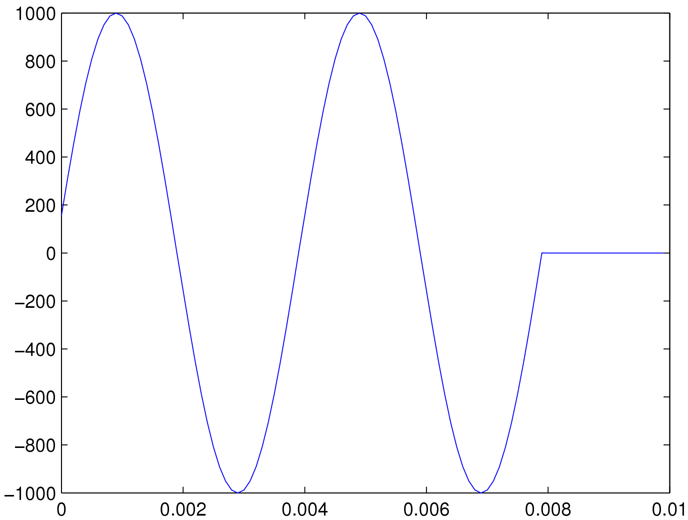


**Fig. 3.** The vertical concentrated sine load acting on the outside surface.

where $Y_{cal}$ is the computer-generated response; $\mathrm{std}(Y_{cal})$ is the standard deviation of $Y_{cal}$; $\mathrm{rand}(-1, 1)$ denotes the random number between $-1$ and $+1$; $l_{noise}$ is a parameter which controls the level of the noise contamination.

In order to investigate the effect of measurement error on the accuracy of the estimated values, we consider the case of noise level namely 5%, and the present method is adopted to determine the dynamic forces. By using a similar argument in Section 4.2, the optimal solution obtained by the present method will be compared to those by the Landweber iterative regularization method and the Tikhonov regularization method. The comparison will be made quantitatively by way of the relative estimation error:

$$\tilde{F} = \left| \frac{F_{\mathrm{Real}} - F_{\mathrm{Identified}}}{F_{\mathrm{Real}}} \right|. \tag{5.3}$$

To evaluate the effectiveness of regularization methods mentioned above, five time points are selected, and for each point the identified force will be compared with the corresponding actual force.

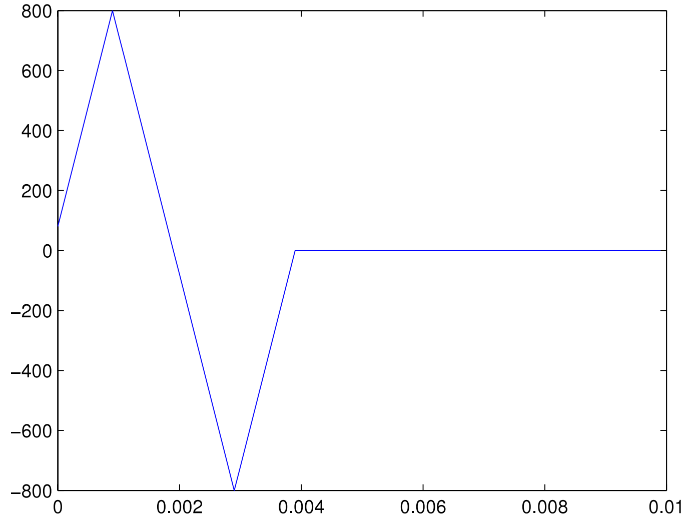


**Fig. 4.** The vertical concentrated triangle load acting on the outside surface.

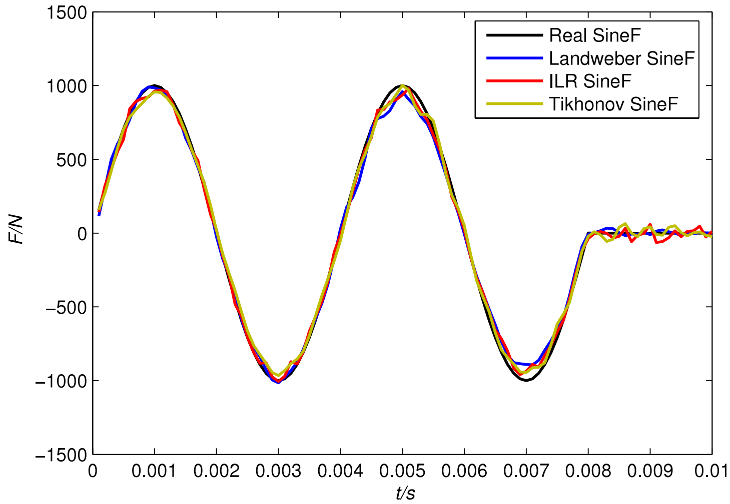


**Fig. 5.** The identified sine force at noise level 5%.

**Table 3**
The identified force at five time points at noise level 5%.

| | Time point | Real force | Landweber iteration regularization method | | Present method | | Tikhonov regularization method | |
|---|---|---|---|---|---|---|---|---|
| | | | Identified force | Error (%) | Identified force | Error (%) | Identified force | Error (%) |
| Sine | 0.001 | 1000 | 984.32 | 1.57 | 968.46 | 1.92 | 956.72 | 4.33 |
| Triangle | 0.0006 | 480 | 450.97 | 3.63 | 488.15 | 1.02 | 416.95 | 4.89 |
| Sine | 0.003 | −1000 | −1015.1 | 1.51 | −1005.1 | 0.51 | −965.72 | 3.43 |
| Triangle | 0.001 | 800 | 698.84 | 12.65 | 736.32 | 7.96 | 730.89 | 8.65 |
| Sine | 0.0045 | 707.11 | 711.23 | 0.41 | 646.27 | 6.08 | 656.19 | 5.09 |
| Triangle | 0.0016 | 320 | 276.32 | 5.46 | 297.67 | 2.79 | 416.07 | 4.90 |
| Sine | 0.0063 | −453.99 | −401.43 | 5.26 | −402.3 | 5.17 | −430.15 | 2.38 |
| Triangle | 0.0033 | −560 | −538.24 | 2.72 | −542.68 | 2.16 | −611.65 | 3.54 |
| Sine | 0.0073 | −891.01 | −793.5 | 9.75 | −909.33 | 1.81 | −878.97 | 1.20 |
| Triangle | 0.0038 | −160 | −149.31 | 1.34 | −157.71 | 0.29 | −213.67 | 6.71 |
| | Error (%) | | Maximum | Average | Maximum | Average | Maximum | Average |
| | Sine | | 12.05 | 3.53 | 9.81 | 3.24 | 9.09 | 3.42 |
| | Triangle | | 14.62 | 5.64 | 10.75 | 3.23 | 11.74 | 3.87 |

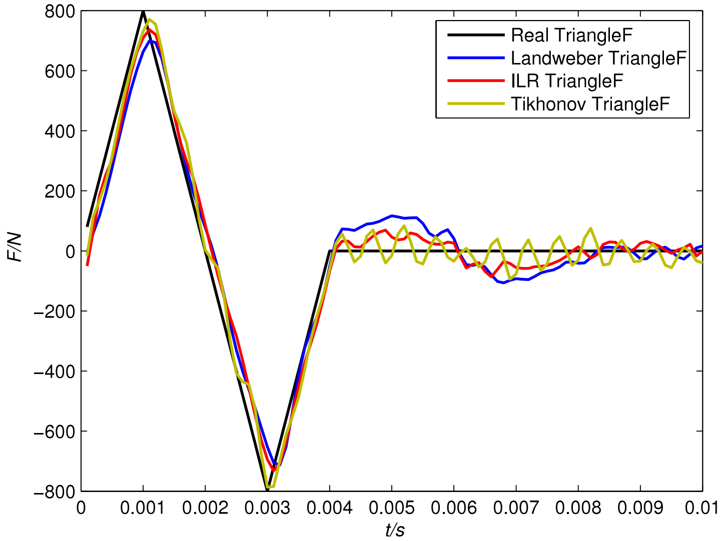


**Fig. 6.** The identified triangle force at noise level 5%.

The results of numerical simulations are as follows:

From Figs. 5–6, it can be shown that three regularization methods can both stably and effectively identify the multi-source dynamic loads by the measured noisy responses. Moreover, the more detailed results by them at five time points are listed in Table 3. It can be found that at these five time points for noise level ±5%, the deviations of the identified loads by the present method are smaller than the Landweber iteration and Tikhonov regularization methods, which are due to better efficient identification. It can be also found that most deviations by the Landweber iteration method, the present method and the Tikhonov regularization method concentrate in the range of 15%, 11%, 12%, respectively. In addition, for the identification of sine force, the maximal deviation and average deviation by the present method are 9.81%, 3.24%, respectively, obviously smaller than the other two. Furthermore, we can find that the maximal deviation and average deviation of the identification of triangle force by the present method are 10.75%, 3.23%, respectively, both smaller than the Landweber iteration method and the Tikhonov regularization method. Meanwhile, the number of iterations by the present method is 12, smaller than the Landweber iteration method, and the optimum asymptotic convergence order of the regularized solution by the present method is better than the Landweber iteration regularization method, which is supported in numerical simulations. In other words, the present algorithm achieves an excellent estimation, and also gives satisfactory results when recovering the loading time function.

## 6. Conclusion

In this paper, an improved iteration regularization method is proposed and considered as an alternative to approximate the solutions of linear ill-posed problems or ill-conditioned matrix equations. It has been found that we can obtain the better optimum asymptotic convergence order of the regularized solution by the present method than the Landweber iteration method and the Tikhonov regularization method. Also, it is validated by numerical example test and suggested to identify the multi-source dynamic loads acting on a plate by the noisy responses. Numerical simulations have shown that the proposed method reduces the number of iterations, quickens the speed of convergence of the regularized solutions, and also demonstrate that the present method is effective and accurate in solving the load identification problems of practical engineering.

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